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The onset of bio-thermal convection in a suspension of gyrotactic microorganisms in a fluid layer with an inclined temperature gradient

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Abstract

Purpose – The purpose of this paper is to investigate a combined bioconvection and thermal instability problem in a horizontal layer of finite depth with a basic temperature gradient inclined to the vertical. The basic flow, driven by the horizontal component of temperature gradient, is the Hadley circulation, which becomes unstable when the vertical temperature difference and density stratification induced by upswimming of microorganisms that are heavier than water become sufficiently large.

Design/methodology/approach – Linear stability analysis of the basic state is performed; the numerical problem is solved using the collocation method.

Findings – The steady-state solution of this problem is obtained. Linear stability analysis of this steady-state solution for the case of three-dimensional disturbances is performed; the numerical problem is solved using the collocation method. The stability problem is governed by three Rayleigh numbers: the bioconvection Rayleigh number and two thermal Rayleigh numbers characterizing temperature gradients in the vertical and horizontal directions, respectively.

Research limitations/implications – Further research should address the application of weakly non-linear analysis to this problem.

Practical implications – The dependence of the critical bioconvection Rayleigh number on the two thermal Rayleigh numbers and other relevant parameters is investigated.

Originality/value – This paper presents what is believed to be the first research dealing with the effect of inclined temperature gradient on the stability of bioconvection in a suspension of gyrotactic microorganisms.

Keywords Convection, Temperature distribution, Microorganisms, Flow
Paper type Research paper

Nomenclature

a	radius of a cell (which is approximated as a spheroid)
B	gyrotactic orientation parameter defined by Equation (14)
D	diffusivity of microorganisms
g	gravity
G	gyrotaxis number, $G = BD/H^2$
h	displacement of the center of gravity from the geometrical center of the cell
H	depth of the horizontal fluid layer

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$\tilde{\mathbf{j}}$	total flux of microorganisms due to macroscopic convection of the fluid, self-propelled swimming of microorganisms, and diffusion of microorganisms, $\tilde{n}\tilde{\mathbf{v}} + nW_c\hat{\mathbf{p}} - D\nabla\tilde{n}$
k, l	dimensionless wavenumbers in the x- and y-directions
$\hat{\mathbf{k}}$	vertically upward unit vector
m	dimensionless wavenumber, $m^2 = k^2 + l^2$
m_c	mass of a microorganism
n	dimensionless number density of motile microorganisms, $\theta\tilde{n}$
$\tilde{n}$	number density of motile microorganisms
$\bar{n}$	dimensionless average number density of microorganisms in the fluid layer
p	excess pressure (above hydrostatic)
$\hat{\mathbf{p}}$	unit vector indicating the direction of microorganisms' swimming
Pe	bioconvection Péclet number, $\mathrm{Pe} = W_cH/D$
Pr	Prandtl number, $\mathrm{Pr} = \mu/\alpha\rho_w$
Ra_B	bioconvection Rayleigh number, $\mathrm{Ra}_B = gH^3\Delta\rho\nu/\mu D$
Ra_H	horizontal thermal Rayleigh number, $\mathrm{Ra}_H = g\beta H^4\beta_H\rho_w/\mu\alpha$
Ra_V	vertical thermal Rayleigh number, $\mathrm{Ra}_V = g\beta H^3\Delta T\rho_w/\mu\alpha$
Sc	Schmidt number, $\mathrm{Sc} = \mu/D\rho_w$
t	dimensionless time, $\tilde{t}\mu/\rho H^2$
$\tilde{t}$	time
T	dimensionless temperature, $\tilde{T} - \tilde{T}_0/\Delta T$
$\tilde{T}$	temperature
$\tilde{T}_0$	ambient temperature
ΔT	temperature difference between the lower and upper surfaces

$\tilde{u}, \tilde{v}, \tilde{w}$ $\tilde{x}, \tilde{y}, \tilde{z}$	velocity components, respectively
U_s, V_s, W_s	x-, y-, z-components of the dimensionless steady-state velocity, $(\tilde{U}_sH\rho/\mu)$, $(\tilde{V}_sH\rho/\mu)$, $(\tilde{W}_sH\rho/\mu)$, respectively
$\tilde{U}_s, \tilde{V}_s, \tilde{W}_s$	$\tilde{x}$-, $\tilde{y}$-, $\tilde{z}$-components of the steady-state velocity, respectively
$\tilde{\mathbf{v}}$	fluid convection velocity vector $(\tilde{u}, v, \tilde{w})$
W_c	average microorganisms' swimming velocity relative to the fluid
$\tilde{x}, \tilde{y}, \tilde{z}$	Cartesian coordinates ($\tilde{z}$ is the upward vertical coordinate)
x, y, z	dimensionless Cartesian coordinates, $\tilde{x}/H$, $\tilde{y}/H$, and $\tilde{z}/H$, respectively

Greek symbols

α	thermal diffusivity of the suspension
β	volumetric expansion coefficient of the fluid
β_H	horizontal temperature gradient
$\Delta\rho$	density difference, $\Delta\rho = \rho_{cell} - \rho_w$
η	parameter defined by Equation (25)
ε	small dimensionless perturbation amplitude
θ	average volume of a microorganism
Θ	rescaled dimensionless amplitude of temperature, $\beta\Delta T(\rho_w/\Delta\rho)\Omega$
μ	dynamic viscosity
ν	integration constant defined by Equation (17)

ξ	parameter defined by Equation (24)
ρ_w	density of water
ρ_{cell}	density of cells
σ	dimensionless disturbance frequency
Φ	dimensionless amplitude of the vertical disturbance velocity, $\nu\,\mathrm{PeSc}W$
Ψ	dimensionless amplitude of the horizontal disturbance velocity, $\nu\,\mathrm{PeSc}U$
ϑ	angle between vector $\hat{\mathbf{p}}$ and the positive direction of the vertical axis z
ϕ	angle between vector $\hat{\mathbf{p}}$ and the positive direction of the horizontal axis x

Subscript

s	steady-state

Superscript

$*$	perturbation value

1. Introduction

Bioconvection is the macroscopic fluid motion resulting from a complex interaction of phenomena characterized by different length scales. The process is driven by the upswimming of self-propelled microorganisms that are denser than water; the upswimming of each microorganism is a mesoscale phenomenon. The macroscopic density gradient caused by this upswimming induces convection instability that results in the formation of periodic falling plumes in the fluid. The theory of bioconvection was introduced by Childress *et al.* (1975); significant progress in stability analysis and numerical investigation of bioconvection patterns caused by gyrotactic microorganisms was made in Pedley *et al.* (1988), Hill *et al.* (1989), Pedley and Kessler (1992), and Ghorai and Hill (1999, 2000).

As explained in Hill and Bees (2002), many species of algae tend to swim upwards in otherwise still water because they are bottom-heavy. Once bioconvection develops, the local shear flow imposes a viscous torque on the cell tipping it away from the vertical, which in turn generates a counterbalancing gravitational torque (because the alga cell is bottom-heavy). These two competing torques impose a bias on the cell's random motion; this behavior was termed gyrotaxis by Kessler (1984).

Recently, Bearon and Grünbaum (2006) extended the theory of bioconvection by considering the environment with salinity stratification; they investigated bioconvection in a deep chamber with a stable linear salinity gradient. Kuznetsov (2005a, b, c, 2006a, b) and Nield and Kuznetsov (2006) introduced the theory of bio-thermal convection. Alloui *et al.* (2005) numerically investigated bioconvection of gravitactic microorganisms in a vertical cylinder, Alloui *et al.* (2006, 2007) presented numerical simulations and linear stability analysis of bio-thermal convection associated with heating or cooling from below. Ghorai and Hill (2007) extended the analysis of bioconvection plumes to three dimensions; they studied the structure and stability of a three-dimensional plume in a deep rectangular box with stress-free sidewalls.

Weber (1973, 1978) pioneered the investigation of the stability problem for convection induced by the inclined temperature gradient. Nield (1994a) reformulated the linear stability analysis for the inclined temperature gradient problem to allow for any value of the Prandtl number. Recent advances in convection problem associated with an inclined temperature gradient are documented in Kaloni and Lou (2002, 2005), who considered extensions of this problem to Oldroyd-B and viscoelastic fluids; and

Shklyaev and Nepomnyashchy (2004), who studied the stability of thermocapillary flows generated by an inclined temperature gradient.

The purpose of this study is to investigate a combined bioconvection and thermal instability problem in a horizontal layer of finite depth with a basic temperature gradient inclined to the vertical, so that the basic flow (driven by the horizontal component of temperature gradient) is a single cell – the Hadley circulation. A similar problem for the suspension of oxytactic motile microorganisms has been recently investigated in Avramenko and Kuznetsov (n.d.). The purpose of this paper is to extend the research of Avramenko and Kuznetsov (n.d.) to investigate the case of gyrotactic microorganisms, such as many species of algae.

2. Governing equations

The model for bioconvection used in this study is based on that presented in Pedley *et al.* (1988) and Hill *et al.* (1989). This model is supplemented by an energy equation and a buoyancy term in the momentum equation that results from the inclined temperature gradient. It is assumed that heating due to the inclined temperature gradient is sufficiently weak, so it does not kill microorganisms and does not affect their gyrotactic behavior. The geometry of the problem is similar to that considered in Nield (1994a) (see Figure 1). The suspension of gyrotactic microorganisms is confined in a shallow horizontal box with small height-to length and height-to-width aspect ratios. The two horizontal walls of the box are at a distance H apart. A Cartesian coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ (with the $\tilde{z}$-axis vertically upward) is chosen such that its origin is in the middle of the box. A linear horizontal temperature gradient is imposed in the $\tilde{x}$-direction, and a constant temperature difference is imposed between the two horizontal walls. The situation described above can be realized in a carefully planned lab experiment. The Boussinesq approximation is utilized. Under these assumptions, the governing equations can be presented as

$$\rho_w\left(\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}\right) = -\nabla p + \mu\nabla^2\tilde{\mathbf{v}} + \hat{\mathbf{k}}(n\theta\Delta\rho g - \rho_w g\beta(\tilde{T} - \tilde{T}_0)) \qquad (1)$$

$$\operatorname{div}(\tilde{\mathbf{v}}) = 0 \qquad (2)$$

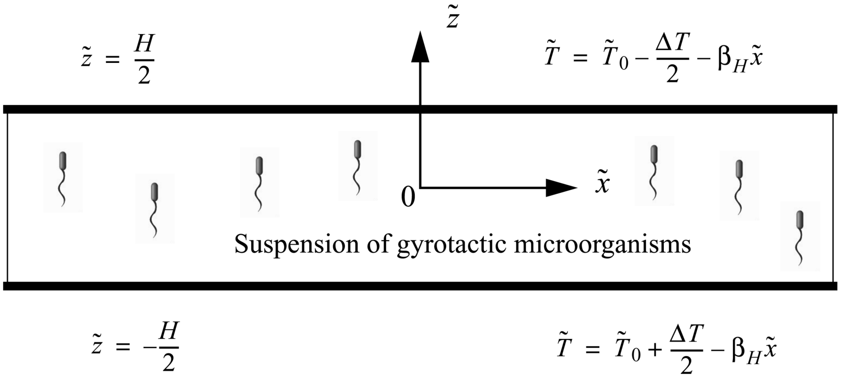

Figure 1.
Definition sketch

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{n} = -\mathrm{div}(\tilde{n} W_c \hat{\mathbf{p}} - D\nabla \tilde{n}) \tag{3}$$

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{T} = \alpha \nabla \tilde{T} \tag{4}$$

where D is the diffusivity of microorganisms (this assumes that all random motions of microorganisms can be approximated by a diffusive process); α is the thermal diffusivity of the suspension (assumed to be approximately the same as that of water); g is the gravity; $\tilde{n}$ is the number density of motile microorganisms; $\tilde{T}$ is the temperature; $\tilde{T}_0$ is the ambient temperature; p is the excess pressure (above hydrostatic); $\hat{\mathbf{k}}$ is the vertically upward unit vector, $\hat{\mathbf{p}}$ is the unit vector indicating the direction of microorganisms' swimming; $\tilde{t}$ is the time; $\tilde{\mathbf{v}}$ is the fluid convection velocity vector with components $\tilde{u}$, $\tilde{v}$, $\tilde{w}$; $W_c\hat{\mathbf{p}}$ is the vector of average swimming velocity relative to the fluid (W_c is assumed to be constant); θ is the average volume of a microorganism; μ is the dynamic viscosity, assumed to be approximately the same as that of water; $\Delta\rho = \rho_{cell} - \rho_w$ is the density difference; ρ_w is the density of water; and ρ_{cell} is the density of cells.

3. Boundary conditions

As in Nield (1994a), the ratio of the height to the length of the layer is assumed to be sufficiently small so that fluid motion in the horizontally central part is not affected by lateral end effects.

At the bottom of the layer (assumed to be rigid, impermeable to microorganisms, and at a constant and uniform temperature), the following conditions are satisfied:

$$\tilde{\mathbf{v}} = 0, \quad \tilde{T} = \tilde{T}_0 + \frac{\Delta T}{2} - \beta_H \tilde{x}, \quad \tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \quad \text{at } \tilde{z} = -\frac{H}{2} \tag{5}$$

where β_H is the constant horizontal temperature gradient and $\tilde{\mathbf{j}} = \tilde{n}\tilde{\mathbf{v}} + nW_c\hat{\mathbf{p}} - D\nabla\tilde{n}$ is the total flux of microorganisms due to macroscopic convection of the fluid, self-propelled swimming of microorganisms, and diffusion of microorganisms.

The upper surface of the layer is assumed rigid as well because, according to Hill *et al.* (1989), even if it is open to the air, microorganisms tend to collect at the surface forming what appears to be a packed layer, and it is unlikely that the upper boundary is ever fully stress free. The upper boundary is also assumed impermeable to microorganisms and at a uniform temperature. Under these assumptions, the boundary conditions at the upper surface of the layer are:

$$\tilde{\mathbf{v}} = 0, \quad \tilde{T} = \tilde{T}_0 - \frac{\Delta T}{2} - \beta_H \tilde{x}, \quad \tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \quad \text{at } \tilde{z} = \frac{H}{2} \tag{6}$$

4. Steady-state solution

From Equations (1)-(4) and the fact that the layer is infinite in the horizontal directions it follows that at steady state the number density of microorganisms, $\tilde{n}_s$, is a function of the vertical coordinate only; under this condition the steady-state solutions for the temperature and velocity components are, respectively (Nield, 1994a):

$$T_s = -z - \frac{\mathrm{Ra}_H^2}{\mathrm{Ra}_V}x + \frac{\mathrm{Ra}_H^2}{5,760\mathrm{Ra}_V}(7z - 40z^3 + 48z^5) \tag{7}$$

$$U_s = \frac{\mathrm{Ra}_H}{24\mathrm{Pr}}(z - 4z^3), \quad V_s = W_s = 0 \tag{8}$$

where the dimensionless coordinates, steady-state temperature, and velocity components are defined by the following equations, respectively:

$$z = \tilde{z}/H, \quad x = \tilde{x}/H, \quad T_s = \frac{\tilde{T}_s - \tilde{T}_0}{\Delta T}, \quad U_s = \frac{\tilde{U}_s H\rho}{\mu}, \quad V_s = \frac{\tilde{V}_s H\rho}{\mu}, \quad W_s = \frac{\tilde{W}_s H\rho}{\mu} \tag{9}$$

and the vertical thermal Rayleigh number, horizontal thermal Rayleigh number, and Prandtl number are defined, respectively, as

$$\mathrm{Ra}_V = \frac{g\beta H^3 \Delta T \rho_w}{\mu\alpha}, \quad \mathrm{Ra}_H = \frac{g\beta H^4 \beta_H \rho_w}{\mu\alpha}, \quad \mathrm{Pr} = \frac{\mu}{\alpha\rho_w} \tag{10}$$

The steady solution for the number density of microorganisms is obtained from the following equation (which follows from Equation (3)):

$$n_s \mathrm{Pe} \mathrm{p}_z - \frac{\partial n_s}{\partial z} = 0 \tag{11}$$

where $\mathrm{Pe} = W_c H/D$ is the bioconvection Péclet number, $n_s = \theta\tilde{n}_s$ and $\hat{\mathrm{p}}_z$ is the z-component of vector $\hat{\mathbf{p}}_s$. Using the results obtained in Ghorai and Hill (1999), at steady-state $\hat{\mathrm{p}}_z$ is given by

$$\hat{\mathrm{p}}_z = \begin{cases} \sqrt{1 - \left(G\mathrm{Sc}\dfrac{\partial U_s}{\partial z}\right)^2}, & \left|G\mathrm{Sc}\dfrac{\partial U_s}{\partial z}\right| \leq 1 \\ 0, & \left|G\mathrm{Sc}\dfrac{\partial U_s}{\partial z}\right| > 1 \end{cases} \tag{12}$$

where

$$G = \frac{BD}{H^2}, \quad \mathrm{Sc} = \frac{\mu}{D\rho_w} \tag{13}$$

are the gyrotaxis and Schmidt numbers, respectively, and

$$B = \frac{4\pi\mu a^3}{m_c g h} \tag{14}$$

is the gyrotactic orientation parameter introduced in Pedley and Kessler (1987) (it quantifies the time scale for the reorientation of the microorganisms by the gravitational torque against viscous resistance), where m_c is the mass of a microorganism, a is the radius of a cell (which is approximated as a spheroid; this is a good approximation, for example for many algal cells such as *Chlamydomonas*, Ghorai and Hill, 1999), and h is the displacement of the center of gravity from the geometrical center of the cell.

Accounting for Equations (8) and (12), the solution of Equation (11) is

$$n_s(z) = \nu \exp\left(\mathrm{Pe}\int_{-1/2}^{z} \hat{\mathrm{p}}_z(\hat{z})\mathrm{d}\hat{z}\right) \tag{15}$$

Since under realistic conditions the horizontal temperature gradient is expected to be small (otherwise the temperature will kill microorganisms), the vorticity of the steady-state horizontal flow induced by this temperature gradient is also expected to be small and the condition $|G\mathrm{Sc}(\partial U_s/\partial z)| \leq 1$ is expected to hold in Equation (12). In this case the integral on the left-hand side of Equation (15) is expressed through elliptic integrals (see Equation (A1) in the Appendix).

The integration constant ν in Equation (15) is related to the dimensionless average concentration of microorganisms in the fluid layer, $\bar{n}$, as

$$\bar{n} = \int_{-1/2}^{1/2} n_s(z)\mathrm{d}z = \nu \int_{-1/2}^{1/2} \exp\left(\mathrm{Pe}\int_{-1/2}^{\eta} \hat{\mathrm{p}}_z(z)\mathrm{d}z\right)\mathrm{d}z \tag{16}$$

The integration constant is then given by

$$\nu = \frac{\bar{n}}{\displaystyle\int_{-1/2}^{1/2} \exp\left(\mathrm{Pe}\int_{-1/2}^{z} \hat{\mathrm{p}}_z(z)\mathrm{d}z\right)\mathrm{d}z} \tag{17}$$

Figure 2a displays the vertical projection of the unit vector $\hat{\mathbf{p}}$ indicating the direction of microorganisms' swimming for different values of parameter S^2 (parameter S is defined in Equation (A2) in the Appendix). It is evident that for no horizontal temperature gradient ($S^2 = 0$) at steady-state (zero flow velocity, zero vorticity) microorganisms swim strictly vertically. The horizontal temperature gradient results in steady-state flow changing to Hadley circulation (given by Equation (8)), the non-zero vorticity of the basic flow reorients microorganisms' swimming direction; this decreases the vertical projection of vector $\hat{\mathbf{p}}$. Figure 2a also shows that microorganisms swimming direction is not constant; it depends on the vertical position. At two locations (approximately at $z = \pm 0.29$) the microorganisms swim in the strictly vertical direction independent of the value of parameter S. The largest deviation from vertical upswimming is observed at the bottom and at the top of the fluid layer (at $z = \pm 0.5$ at $S^2 = 0.25$ the swimming direction becomes almost horizontal); also, some significant deviation from vertical occurs in the center of the layer, at $z = 0$. These positions correspond to locations where microorganisms' swimming direction is reoriented most markedly by the viscous torque imposed by the local shear flow.

Figure 2b displays steady-state distributions of microorganisms' number density. For $S^2 = 0$ the number density of microorganisms at the top of the layer (at $z = 0.5$) is

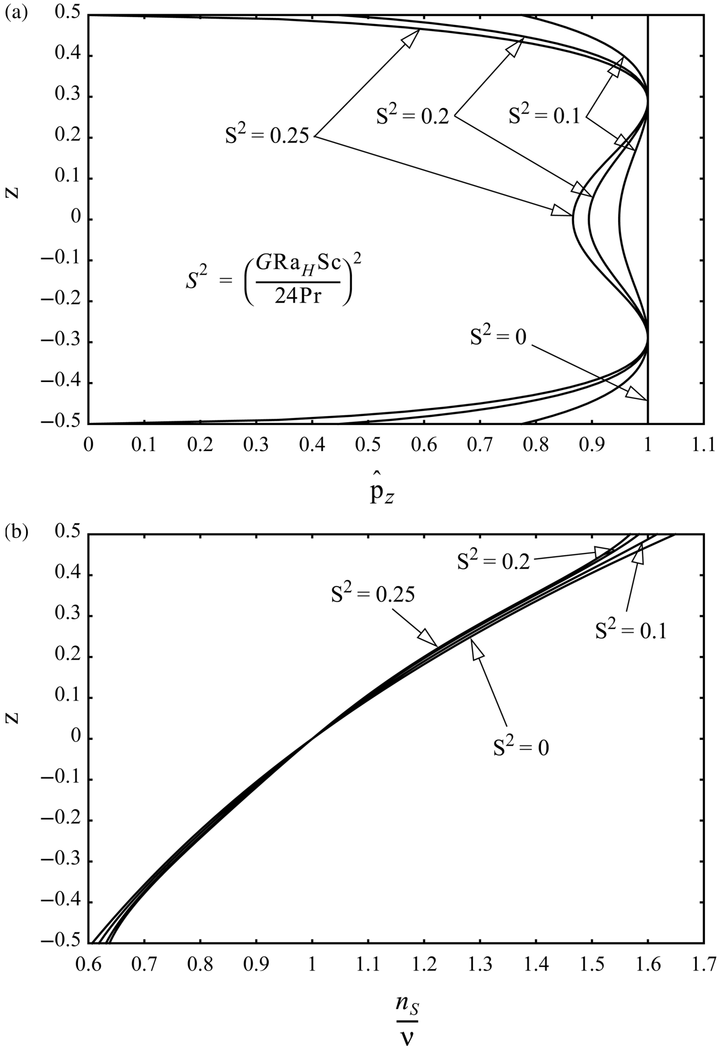

Figure 2.
Effect of the vertical position in the layer at steady state on the vertical projection of the unit vector $\hat{\mathbf{p}}$ indicating the direction of microorganisms' swimming (a) and on distributions of microorganisms' number density (b)

the largest. Since the horizontal steady-state flow results in a non-zero vorticity that reorients the direction of swimming of microorganisms from strictly vertical, the magnitude of microorganisms' vertical velocity component resulting from their self-propelled motion becomes smaller as S^2 increases; this reduces the concentration of microorganisms in the upper portion of the fluid layer. This in turn decreases the density gradient and makes the suspension more stable. This prediction is confirmed later on by the stability analysis (see the discussion of Table III in Results and discussion concerning the effect of Ra_H).

5. Linear stability analysis

The perturbations are introduced as follows:

$$\begin{aligned}[n, T, \mathbf{v}, p, \hat{\mathbf{p}}] = {} & [n_s(z), T_s(x,z), \mathbf{v}_s(U_s(z), 0, 0), p_s(z), \hat{\mathbf{p}}_s(z)] \\ & + \varepsilon[n^*(t,x,y,z), T^*(t,x,y,z), \mathbf{v}^*(u^*(t,x,y,z), \\ & \mathrm{v}^*(t,x,y,z), w^*(t,x,y,z)), P^*(t,x,y,z), \hat{\mathbf{p}}^*(t,x,y,z)]\end{aligned} \tag{18}$$

where ε is the small dimensionless perturbation amplitude, $y = \tilde{y}/H$, and

$$t = \frac{\tilde{t}\mu}{\rho H^2} \tag{19}$$

is the dimensionless time.

According to Pedley *et al.* (1988):

$$\hat{\mathbf{p}} = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta) \tag{20}$$

where ϑ is the angle between vector $\hat{\mathbf{p}}$ and the positive direction of the vertical axis z and ϕ is the angle between vector $\hat{\mathbf{p}}$ and the positive direction of the horizontal axis x. Angle ϑ is represented as

$$\vartheta = \vartheta_0(z) + \varepsilon\vartheta^*(t,x,y,z) \tag{21}$$

where ϑ_0 is the angle between the vector of microorganisms' average swimming velocity and the positive direction of the vertical axis z for the case with no bioconvection (if horizontal temperature gradient is absent, as in the situation considered in Pedley *et al.* (1988), $\vartheta_0 = 0$) and ϑ^* represents the difference between the actual swimming direction and the swimming direction without bioconvection (thus ϑ^* is the perturbation in the swimming direction induced by bioconvection).

Using Equation (21), Equation (20) is recast as follows:

$$\hat{\mathbf{p}}^* = (\vartheta^*\cos\vartheta_0\cos\phi, \vartheta^*\cos\vartheta_0\sin\phi, -\vartheta^*\sin\vartheta_0) \tag{22}$$

Following Pedley *et al.* (1988) and eliminating ϕ the equation for $\hat{\mathbf{p}}^*$ is recast as

$$\hat{\mathbf{p}}^* = G\mathrm{Sc}(\eta\cos\vartheta_0, -\xi\cos\vartheta_0, -(\eta^2+\xi^2)^{1/2}\sin\vartheta_0) \tag{23}$$

where, for a spherical cell,

$$\xi = \frac{\partial w^*}{\partial y} - \frac{\partial \mathrm{v}^*}{\partial z} \tag{24}$$

$$\eta = -\frac{\partial w^*}{\partial x} + \frac{\partial u^*}{\partial z} \tag{25}$$

As one can see from (23), the z-component of vector $\hat{\mathbf{p}}^*$ depends non-linearly on the perturbation parameters ξ and η. Expanding the square root $(\eta^2+\xi^2)^{1/2}$ in Taylor series leads to the following approximation:

$$
\begin{aligned}
f(\eta,\xi) = (\eta^2+\xi^2)^{1/2} &= f(0,0) + \left.\frac{\partial f}{\partial \eta}\right|_{(\eta=0,\xi=0)} \eta + \left.\frac{\partial f}{\partial \xi}\right|_{(\eta=0,\xi=0)} \xi + O(\eta^2,\xi^2) \\
&= \left(\frac{\eta}{\sqrt{\eta^2+\xi^2}}\right)_{(\eta=0,\xi=0)} \eta + \left(\frac{\xi}{\sqrt{\eta^2+\xi^2}}\right)_{(\eta=0,\xi=0)} \xi + O(\eta^2,\xi^2)
\end{aligned} \tag{26}
$$

The determination of coefficients in the linear terms in Equation (26) requires obtaining the following limits:

$$
\lim_{\substack{\eta\to 0\\ \xi\to 0}} \frac{\eta}{\sqrt{\eta^2+\xi^2}} \quad \text{and} \quad \lim_{\substack{\eta\to 0\\ \xi\to 0}} \frac{\xi}{\sqrt{\eta^2+\xi^2}} \tag{27}
$$

In order to calculate these limits, it is assumed that η and ξ approach zero at the same rate; in this case Equation (23) can be recast as

$$
\hat{\mathbf{p}}^* = G\mathrm{Sc}\left(\eta\cos\vartheta_0, -\xi\cos\vartheta_0, -(\eta+\xi)\sin\vartheta_0/\sqrt{2}\right) \tag{28}
$$

The substitution of Equations (18) into the dimensionless version of Equations (1)–(4) and linearizing results in the following equations for perturbations:

$$
\frac{\partial u^*}{\partial t} + U_s\frac{\partial u^*}{\partial x} + w^*\frac{\partial U_s}{\partial z} = -\frac{\partial p^*}{\partial x} + \nabla^2 u^* \tag{29}
$$

$$
\frac{\partial v^*}{\partial t} + U_s\frac{\partial v^*}{\partial x} = -\frac{\partial p^*}{\partial y} + \nabla^2 v^* \tag{30}
$$

$$
\frac{\partial w^*}{\partial t} + U_s\frac{\partial w^*}{\partial x} = -\frac{\partial p^*}{\partial z} + \nabla^2 w^* - \frac{\mathrm{Ra}_B}{\mathrm{Sc}}\frac{n^*}{\nu} + \frac{\mathrm{Ra}_V}{\mathrm{Pr}}T^* \tag{31}
$$

$$
\nabla \mathbf{v}^* = 0 \tag{32}
$$

$$
\frac{\partial n^*}{\partial t} + U_s\frac{\partial n^*}{\partial x} + w^*\frac{\partial n_s}{\partial z} = -\mathrm{div}[n_s\mathrm{Pe}\hat{\mathbf{p}}^* + n^*\mathrm{Pe}\hat{\mathbf{p}}_s - \mathrm{Sc}^{-1}\nabla n^*] \tag{33}
$$

$$
\frac{\partial T^*}{\partial t} + U_s\frac{\partial T^*}{\partial x} + w^*\frac{\partial T_s}{\partial z} = \mathrm{Pr}^{-1}\nabla^2 T^* \tag{34}
$$

where

$$
\mathrm{Ra}_B = \frac{gH^3\Delta\rho\nu}{\mu D} \tag{35}
$$

The stability of linear differential Equations (29)–(34) is examined in terms of individual Fourier modes:

$$
\begin{aligned}
&[n^*(t,x,y,z), T^*(t,x,y,z), \mathbf{v}^*(u^*(t,x,y,z), \mathrm{v}^*(t,x,y,z), w^*(t,x,y,z)), P^*(t,x,y,z)] \\
&\quad = [N(z), \Omega(z), \mathbf{V}(U(z), V(z), W(z)), P(z)]\exp[i(kx+ly-\sigma t)]
\end{aligned} \tag{36}
$$

where k and l are the dimensionless wavenumbers in the x- and y-directions, and σ is the dimensionless disturbance frequency. Substituting Equations (36) into Equations (29)–(34) results in equations for the amplitudes U, V, W, P, N and Ω. The elimination of P and V from these equations for the amplitudes and the utilization of Equations (7), (8), and (15) gives the following equations for the remaining amplitudes Φ (Φ is related to W by the first equation in (41)), Ψ (Ψ is related to U by the second equation in (41)), N, and Θ (Θ is related to Ω by the third equation in (41)):

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\frac{m^2}{k^2-l^2}\left(m^2+i(kU_s-\sigma)\right)\right]\Psi\\&\quad=\frac{1}{k^2-l^2}\left[ik\frac{\mathrm{d}^3}{\mathrm{d}z^3}-i\left(k^3+k\left(l^2+i(kU_s-\sigma)\right)\right)\frac{\mathrm{d}}{\mathrm{d}z}+l^2\frac{\mathrm{d}U_s}{dz}\right]\Phi\end{aligned}\tag{37}$$

$$\begin{aligned}&\left[\frac{\mathrm{d}^4}{\mathrm{d}z^4}-\left[2m^2+i(kU_s-\sigma)\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right.\\&\quad\left.+\left[m^4+i\left(m^2kU_s+k\frac{\mathrm{d}^2U_s}{\mathrm{d}z^2}-m^2\sigma\right)\right]\right]\Phi=m^2\mathrm{Ra}_B\mathrm{Pe}(\Theta-N)\end{aligned}\tag{38}$$

$$\begin{aligned}&l\exp\left(-\mathrm{Pe}\int_{-1/2}^{z}\hat{\mathrm{p}}_z\mathrm{d}z\right)\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\mathrm{Pe}\frac{\mathrm{d}}{\mathrm{d}z}-\left(m^2+i\mathrm{Sc}(kU_s-\sigma)\right)\right]N\\&=\left[l\left(\hat{\mathrm{p}}_z(1+Gm^2)-i\frac{k-l}{\sqrt{2}}G\frac{\mathrm{dp}_z}{\mathrm{d}z}\frac{2\hat{\mathrm{p}}_z^2-1}{\sqrt{1-\hat{\mathrm{p}}_z^2}}\right)+il\frac{k-l}{\sqrt{2}}G\hat{\mathrm{p}}_z(1-\hat{\mathrm{p}}_z^2)\frac{\mathrm{d}}{\mathrm{d}z}\right.\\&\quad\left.-\left(lG\hat{\mathrm{p}}_z+\frac{i}{\sqrt{2}}G\frac{\mathrm{dp}_z}{\mathrm{d}z}\frac{2\hat{\mathrm{p}}_z^2-1}{\sqrt{1-\hat{\mathrm{p}}_z^2}}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{i}{\sqrt{2}}G\hat{\mathrm{p}}_z(1-\hat{\mathrm{p}}_z^2)\frac{\mathrm{d}^3}{\mathrm{d}z^3}\right]\Phi\\&\quad+\frac{k+l}{\sqrt{2}}G\left[\frac{\mathrm{dp}_z}{\mathrm{d}z}\frac{2\hat{\mathrm{p}}_z^2-1}{\sqrt{1-\hat{\mathrm{p}}_z^2}}\frac{\mathrm{d}}{\mathrm{d}z}-\hat{\mathrm{p}}_z(1-\hat{\mathrm{p}}_z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right]\Psi\end{aligned}\tag{39}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\left(m^2+i\mathrm{Pr}(kU_s-\sigma)\right)\right]\Theta=\frac{\mathrm{Ra}_V}{\mathrm{Ra}_B\mathrm{Pe}}\frac{\mathrm{d}T_s}{\mathrm{d}z}\Phi\tag{40}$$

where

$$\Phi=\nu\mathrm{PeSc}W,\quad \Psi=\nu\mathrm{PeSc}U,\quad \Theta=\beta\Delta T\frac{\rho_w}{\Delta\rho}\Omega,\quad m^2=k^2+l^2\tag{41}$$

Since both the lower and upper boundaries of the layer are assumed rigid, Equations

(37)–(40) must be solved subject to the following boundary conditions:

$$\Psi = 0, \quad \Phi = 0, \quad \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0, \quad \Theta = 0, \quad \mathrm{Pe}N = \frac{\mathrm{d}N}{\mathrm{d}z} \quad \text{at } z = -\frac{1}{2} \tag{42}$$

$$\Psi = 0, \quad \Phi = 0, \quad \frac{\mathrm{d}\Phi}{\mathrm{d}z} = 0, \quad \Theta = 0, \quad \mathrm{Pe}N = \frac{\mathrm{d}N}{\mathrm{d}z} \quad \text{at } z = -\frac{1}{2} \tag{43}$$

6. Numerical method

The collocation method (Fletcher, 1984) is used for obtaining the numerical solution of the eigenvalue problem for Equations (37)-(40). To estimate the numerical inaccuracy of the method two different sets of basis functions (satisfying boundary conditions given by Equations (42) and (43)) are utilized. The first set of the basis functions is:

$$\Phi = \sum_{j=1}^{M} a_j \left(z^2 - \frac{1}{4}\right)^{2j}, \quad \Psi = \sum_{j=1}^{M} b_j z \left(z^2 - \frac{1}{4}\right)^{2j-1}, \quad \Theta = \sum_{j=1}^{M} c_j \left(z^2 - \frac{1}{4}\right)^{j},$$

$$N = \sum_{j=1}^{M} d_j \left[2 - \frac{2(\mathrm{Pe}+\mathrm{Pe}^2)}{2+2\mathrm{Pe}+\mathrm{Pe}^2}\left(1 - \left(z - \frac{1}{2}\right)\right) - \frac{2\mathrm{Pe}^2}{2+2\mathrm{Pe}+\mathrm{Pe}^2}\left(\left(z - \frac{1}{2}\right) - \left(z - \frac{1}{2}\right)^2\right)\right]^j \tag{44}$$

The second set of the basis functions is:

$$\Phi = \sum_{j=1}^{M} a_j \left(z^2 - \frac{1}{4}\right)^{2j}, \quad \Psi = \sum_{j=1}^{M} b_j z \left(z^2 - \frac{1}{4}\right)^{2j-1}, \quad \Theta = \sum_{j=1}^{M} c_j \left(z^2 - \frac{1}{4}\right)^{j},$$

$$N = \sum_{j=1}^{M} c_j \left[2 - \frac{2(\mathrm{Pe}+\mathrm{Pe}^2)}{2+3\mathrm{Pe}+2\mathrm{Pe}^2}\left(1 - \left(z - \frac{1}{2}\right)\right) - \frac{2\mathrm{Pe}^2}{2+3\mathrm{Pe}+2\mathrm{Pe}^2}\left(1 - \left(z - \frac{1}{2}\right)^2\right)\right]^j \tag{45}$$

The solution of the eigenvalue problem leads to the following dependence for the bioconvection Rayleigh number:

$$\mathrm{Ra}_B = \mathrm{Ra}_B(k, l, \mathrm{Ra}_V, \mathrm{Ra}_H, \mathrm{Pe}, G, \mathrm{Sc}, \mathrm{Pr}, \sigma) \tag{46}$$

A neutral stability curve is defined as the locus of points on which the imaginary part of σ is equal to zero, where σ is defined in Equation (36). If the real part of σ is equal to zero on the stability curve, the instability is stationary, otherwise it is oscillatory. Oscillatory instability usually occurs when the instability is caused by two or more competing processes, one of which is pushing the system away from the basic state and the other one is trying to return the system back to the basic state (Chandrasekhar, 1961). Negative values of Ra_V (as in some cases presented in Table III below) correspond to negative ΔT, that is when the upper plate is at a higher temperature than the lower plate. Without bioconvection, the layer with negative ΔT would never become unstable, but upswimming of microorganisms results in two competing effects in this case: the unstable density stratification due to microorganism's upswimming and stable density stratification due to temperature gradient. It is shown in Nield and Kuznetsov (2006) that such situation may lead to oscillatory instability. However, for all parameter values utilized in this paper instability is found to be monotonic, that is the real part of σ is equal to zero on the stability curve. It remains the subject of further investigation to analyze whether oscillatory instability is possible at all for this particular flow situation.

The critical bioconvection Rayleigh number is then obtained as

$$\mathrm{Ra}_{B,cr} = \min\{\mathrm{Ra}_B(k, l, \mathrm{Ra}_V, \mathrm{Ra}_H, \mathrm{Pe}, G, \mathrm{Sc}, \mathrm{Pr}, \sigma_i = 0)\} \tag{47}$$

where σ_i is the imaginary part of σ.

Computational results indicate that the minimum of Ra_B is attained when $k \to l$ (which is equivalent to $\delta = k - l \to 0$). If $k \to l$, the coefficient by the senior derivative in Equation (37) approaches zero and Equation (37) becomes singular. To overcome this singularity the standard perturbation technique presented in Kevorkian and Cole (1985) is utilized. According to this technique, the inner, z/δ, and outer, z, variables are introduced. The solutions obtained for the inner and outer domains are then matched. The functions given by Equations (44) and (45) are used as initial approximations in the numerical implementation of the method.

Test calculations have shown that for $M = 200$ the difference in $\mathrm{Ra}_{B,cr}$ resulting from using different sets of basis functions (given by Equations (44) and (45), respectively), is found to be less than 0.3 percent for various combinations of dimensionless parameters in Equation (46).

7. Results and discussion

Typical values of physical parameters for the alga *Chlamydomonas nivalis* are given in Table II of Hill *et al.* (1989) and in Table I of Ghorai and Hill (2000). Table II of Pedley *et al.* (1988) presents values for *C. nivalis* as well as typical parameter ranges for all gyrotactic microorganisms based on data presented in literature (Kessler, 1986a) and on observations. In order to estimate dimensionless parameters, the depth of the layer is assumed to be 6 mm, as in Kessler (1986b). Suspension properties such as density, viscosity, thermal diffusivity, and thermal expansion coefficient are assumed to be approximately the same as those for water and taken from tables in Bejan (1995). For estimating the vertical and horizontal temperature gradients a 10 °C temperature difference in both vertical and horizontal directions is assumed; the characteristic length in the horizontal direction is assumed to be 100 mm. The resulting ranges of dimensionless parameters are summarized in Table I. The values of Ra_V and Ra_H given in Table I should be understood as estimates of the maximum safe values of these parameters so that the temperature variation does not kill microorganisms; smaller values of Ra_V and Ra_H can be easily realized in a lab experiment. The value of Ra_B can be also varied from that given in Table I by changing the average concentration of microorganisms.

All obtained computational results indicate that the minimum of the function Ra_B (as defined by Equation (47)) occurs at $k \to l\,(\delta \to 0)$. For all cases the minimum of the function Ra_B lies in the following range of wavenumbers: $k = l = 0.13\text{-}2.24$. This means that the minimum is attained not for a single wavenumber m_{cr} (m is defined by third equation in (41)), but for a range of wavenumbers for which $k = l$ (the range of k and l is given above). For this reason, the critical wavenumber m_{cr} is not given in Tables II–VI.

Table II shows the effect of the Péclet number. These results are obtained for the following dimensionless parameters values: $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $G = 0.01$, and $\mathrm{Pr} = \mathrm{Sc} = 1$. The results indicate that increasing the bioconvection Péclet number leads to the decrease of the bioconvection Rayleigh number. This means that the system becomes less stable, which is as expected because larger bioconvection Péclet number means that the suspension consists of faster swimming microorganisms.

Table I.
Values of dimensionless parameters

Parameter	Definition	Range or typical value
Sc	$\frac{\mu/\rho_w}{D}$	2-200
Pr	$\frac{\mu/\rho_w}{\alpha}$	2.5-11
Pe	$\frac{W_c H}{D}$	0-20
Ra_V	$\frac{g\beta H^3 \Delta T \rho_w}{\mu\alpha}$	138,000
Ra_H	$\frac{g\beta H^4 \beta_H \rho_w}{\mu\alpha}$	1,380
G	$\frac{BD}{H^2}$	1/Pe^2
Ra_B	$\frac{gH^3 \Delta\rho\nu}{\mu D} = \frac{gH^3}{(\mu/\rho_w)D}\frac{\Delta\rho}{\rho_w}\frac{\bar{n}\,\mathrm{Pe}}{\exp(\mathrm{Pe})-1}$	28,000

Table II.
The effect of the bioconvection Péclet number on the critical bioconvection Rayleigh number

Pe	0.2	0.5	1	2	5
$\mathrm{Ra}_{B,cr}$	8,268	1,738	724	437	316

Notes: $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $G = 0.01$ and $\mathrm{Pr} = \mathrm{Sc} = 1$

Table III.
The effect of the vertical and horizontal Rayleigh numbers on the critical bioconvection Rayleigh number

Ra_V Ra_H		−500	−200	−100	0	100	200	500
0	$\mathrm{Ra}_{B,cr}$	756	749	739	728	715	693	623
100	$\mathrm{Ra}_{B,cr}$	766	757	749	738	724	707	642
200	$\mathrm{Ra}_{B,cr}$	793	782	776	768	759	748	707
500	$\mathrm{Ra}_{B,cr}$	941	940	939	938	937	936	933

Notes: $\mathrm{Pe} = 1$, $G = 0.01$ and $\mathrm{Pr} = \mathrm{Sc} = 1$

Faster swimmers increase the unstable density stratification in the layer; this results in a less stable suspension. The similar effect of Pe is observed in a layer subjected to a vertical temperature gradient alone (Kuznetsov, 2005b; Alloui *et al.*, 2007) in suspensions of oxytactic and negatively gravitactic microorganisms, respectively, which indicates certain similarities between suspensions of microorganisms exhibiting different behaviors (negative gravitaxis, gyrotaxis, and oxytaxis). The critical wavenumber also decreases as Pe increases, which happens because the vortex size of the perturbation flow (which is inversely proportional to the wavenumber) grows with the increase of the Péclet number. This is related to the increasing size of the physical system ($\mathrm{Pe} \sim H$).

Computational results presented in Table III demonstrate the effects of the vertical and horizontal Rayleigh numbers on the bioconvection Rayleigh number. These results are computed for $\mathrm{Pe} = 1$, $G = 0.01$, $\mathrm{Pr} = 1$ and $\mathrm{Sc} = 1$. From Table III it is evident that for all values of Ra_H the critical bioconvection Rayleigh number $\mathrm{Ra}_{B,cr}$ decreases with the increase of Ra_V (when Ra_H is kept constant). A similar trend is observed in Kuznetsov (2005b) and Alloui *et al.* (2006, 2007). This happens because increasing Ra_V corresponds to increasing the temperature difference between the lower and upper plates. This induces an additional destabilizing mechanism (in addition to upswimming of microorganisms) which contributes to unstable density stratification thus making the suspension less stable. Also, the critical bioconvection Rayleigh number $\mathrm{Ra}_{B,cr}$ increases with the increase of Ra_H (when Ra_V is kept constant), which implies that the increase of Ra_H stabilizes the system. This is explained as follows. The effect of increasing Ra_H is to distort the basic temperature profile away from the linear one, which produces the stabilizing effect because the destabilizing negative temperature gradient is decreased in magnitude in the bulk of the fluid (Nield, 1994b). The effect of Ra_Von $\mathrm{Ra}_{B,cr}$ becomes less significant with for large values of Ra_H.

G	0	0.01	0.03	0.05	0.1
$\mathrm{Ra}_{B,cr}$	762	724	661	613	566

Notes: Ra_V=Ra_H = 100, Pe = 1 and Pr = Sc = 1

Table IV.
The effect of the gyrotaxis number on the critical bioconvection Rayleigh number

Sc	0.01	0.1	1	10	100
$\mathrm{Ra}_{B,cr}$	523	571	724	981	1,344

Table V.
The effect of the Schmidt number on the critical bioconvection Rayleigh number

Pr	0.01	0.1	1	10	100
$\mathrm{Ra}_{B,cr}$	1,527	1,056	724	530	547

Table VI.
The effect of the Prandtl number on the critical bioconvection Rayleigh number for positive Ra_V ($\mathrm{Ra}_V = 100$)

Figure 3, which is computed for $\mathrm{Pe} = 1$, $G = 0.01$, $\mathrm{Pr} = 1$ and $\mathrm{Sc} = 1$ (the same parameter values as those used for Table III), depicts the effect of Ra_V on $\mathrm{Ra}_{B,cr}$ for different values of Ra_H. This figure shows that $\mathrm{Ra}_{B,cr}$ decreases as Ra_V increases, which reemphasizes the destabilizing effect of the negative vertical temperature gradient (when the lower plate is at a higher temperature than the upper plate). Also, the increase of Ra_H stabilizes the system; this effect is explained in the previous paragraph. At $\mathrm{Ra}_H = 500$ the curve is almost flat, which means that the critical bioconvection Rayleigh number is almost independent of Ra_V. This happens because at this value of Ra_H the destabilizing effect of the negative temperature gradient is almost balanced by the stabilizing effect of the horizontal temperature gradient.

The effect of the gyrotaxis number G is shown in Table IV; the results are computed for $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $\mathrm{Pe} = 1$ and $\mathrm{Pr} = \mathrm{Sc} = 1$. Increasing G destabilizes the suspension and the critical bioconvection Rayleigh number decreases; a similar trend is observed in Nield and Kuznetsov (2006). As it follows from Equation (12), the gyrotaxis number characterizes the deviation of the microorganism's swimming direction from strictly vertical ($G = 0$ corresponds to negatively gravitactic microorganisms that swim against the gravity). Childress *et al.* (1975) established that an infinite uniform suspension of negatively gravitactic microorganisms ($G = 0$) is stable in the absence of cell concentration stratification. Pedley *et al.* (1988) have shown that under the same conditions a suspension of gyrotactic microorganisms ($G > 0$) is unstable. Hence, gyrotaxis helps the development of convection instability, which is in agreement with what follows from Table IV.

The effect of the Schmidt number on the critical bioconvection Rayleigh number is shown in Table V; the results are computed for $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $\mathrm{Pe} = 1$, $G = 0.01$ and $\mathrm{Pr} = 1$. The increase of the Schmidt number stabilizes the basic flow; the effect is most pronounced for large values of Sc.

The effect of the Prandtl number on the critical bioconvection Rayleigh number is opposite to that of the Schmidt number, as shown in Table VI; the results are computed for $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $\mathrm{Pe} = 1$, $G = 0.01$, and $\mathrm{Sc} = 1$. The increase of the Prandtl

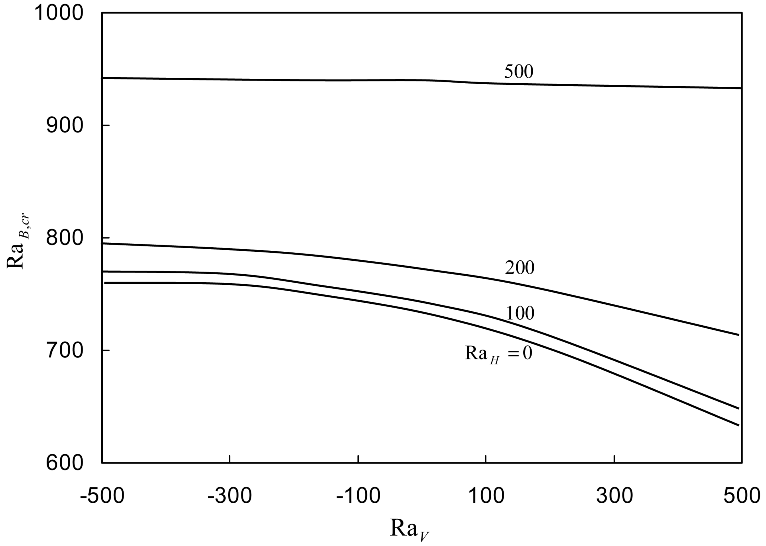

Figure 3.
Effect of the vertical thermal Rayleigh number on the bioconvection Rayleigh number for different values of the horizontal thermal Rayleigh number (computed for $\mathrm{Pe} = 1$, $G = 0.01$, $\mathrm{Pr} = 1$ and $\mathrm{Sc} = 1$)

number destabilizes the basic flow; the effect of the Prandtl number is most pronounced at low values of Pr. The same trend is observed in Nield (1994a), Kaloni and Lou (2002, 2005), and Kaloni and Qiao (1996).

8. Conclusions

Linear stability analysis of the combined bioconvection and thermal instability in a horizontal layer of finite depth with a basic temperature gradient inclined to the vertical is carried out.

Computational results indicate that the minimum of Ra_B is attained when $k \to l$, when the coefficient by the senior derivative in Equation (37) approaches zero and Equation (37) becomes singular. A perturbation method involving matching the inner and outer solutions is utilized to overcome this singularity. For all cases the minimum of the function Ra_B lies in the following range of wavenumbers: $k = l = 0.13\text{-}2.24$.

The increase of the horizontal thermal Rayleigh number (when the vertical thermal Rayleigh number is kept constant) stabilizes the basic flow. The effect of increasing the horizontal thermal Rayleigh number is to distort the basic temperature profile away from the linear one. Also, the critical bioconvection Rayleigh number $\mathrm{Ra}_{B,cr}$ increases with the increase in Ra_V, which implies that the increase of the vertical thermal Rayleigh number stabilizes the system.

It is also established that for a certain value of the vertical thermal Rayleigh number the bioconvection Rayleigh number is independent of the horizontal thermal Rayleigh number. This happens because at this triple point the unstable density distribution caused by upswimming of microorganisms is balanced by the stabilizing effect of the positive temperature gradient (the temperature at the upper plate is higher than the temperature of the lower plate); this diminishes the effect of the horizontal temperature gradient.

The increase of the gyrotaxis number is found to destabilize the suspension. The increase of the Schmidt number stabilizes the basic flow. The increase of the Prandtl number destabilizes the basic flow.

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Appendix

If vorticity of the steady-state horizontal flow induced by the horizontal temperature gradient is small, so that the condition $|G\mathrm{Sc}(\partial U_s/\partial z)| \leq 1$ holds in Equation (12), the steady-state number density of microorganisms given by Equation (15) can be presented as

$$\frac{n_s(z)}{\nu} = \exp\left(\mathrm{Pe}\,\frac{\begin{array}{c} 3z\sqrt{A}[1 - S^2(1-12z^2)^2] + i(1+S) \\ \sqrt{(3+36A)(1-12\left(\dfrac{S}{1+S}\right)z^2)[SE(\phi|s) - F(\phi|s)]} \end{array}}{9\sqrt{A(1-S^2(1-12z^2)^2)}} \right) \qquad (A1)$$

where

$$A = \frac{S}{1-S}z^2, \quad S = \frac{G\mathrm{Ra}_H\mathrm{Sc}}{24\mathrm{Pr}}, \quad \phi = i\,\mathrm{arcsin\,h}\left(2\sqrt{3A}\right), \quad s = -\frac{1-S}{1+S} \qquad (A2)$$

In Equations (A1) and (A2) $E(\phi|s)$ is the elliptic integral of the second kind, $F(\phi|s)$ is the elliptic integral of the first kind, and ν is the integration constant defined by Equation (17).

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